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# Intuitionistic Heuristic Prototype-based Algorithm of Possibilistic Clustering

Dmitri A. Viattchenin
Belarusian State University of Informatics and Radioelectronics

Stanislau Shyrai
Belarusian State University of Informatics and Radioelectronics

## ABSTRACT
This paper introduces a novel intuitionistic fuzzy set-based heuristic algorithm of possibilistic clustering. For the purpose, some remarks on the fuzzy approach to clustering are discussed and a brief review of intuitionistic fuzzy set-based clustering procedures is given, basic concepts of the intuitionistic fuzzy set theory and the intuitionistic fuzzy generalization of the heuristic approach to possibilistic clustering are considered, a general plan of the proposed clustering procedure is described in detail, two illustrative examples confirm good performance of the proposed algorithm, and some preliminary conclusions are formulated.

## General Terms
Pattern Recognition, Clustering, Algorithm.

## Keywords
Intuitionistic Fuzzy Set, Possibilistic Clustering, Allotment among Intuitionistic Fuzzy Clusters, Typical Point.

## 1. INTRODUCTION
Some notes on fuzzy approach to cluster analysis are presented in the first subsection of the section. The second subsection includes a brief review of investigations in the area of intuitionistic fuzzy-set approach to clustering.

### 1.1 Preliminary Remarks
Clustering has long been a popular approach to unsupervised pattern recognition. The goal of clustering is to separate a dataset into self-similar groups such that the objects in the same group have more similarity than the objects in other groups. So, clustering essentially deals with the task of partitioning a set of objects into a number of homogeneous groups, with respect to a suitable similarity measure.

Due to the fuzzy nature of many practical problems, a number of fuzzy clustering methods have been developed following the general fuzzy set theory strategies outlined by Zadeh in his fundamental paper [1]. The main difference between the traditional crisp clustering and fuzzy clustering can be stated as follows. While in crisp clustering an object belongs only to one cluster, in fuzzy clustering objects are allowed to belong to all clusters with different degrees of membership. Heuristic methods, hierarchical methods and objective function-based methods are main approaches in fuzzy clustering.

A possibilistic approach to clustering was proposed by Krishnapuram and Keller [2] and the approach can be considered as a special case of fuzzy approach to clustering because all methods of possibilistic clustering are objective function-based methods. On the other hand, constraints in the possibilistic approach to clustering are less strong than constraints in the fuzzy objective function-based approach to clustering and values of the membership function of a possibilistic partition can be considered as typicality degrees. So, the possibilistic approach to clustering is more general and flexible approach to clustering than the fuzzy approach. Many fuzzy and possibilistic clustering algorithms could be found in the corresponding books [3], [4], [5], [6].

Objective function-based approach in fuzzy clustering is most common and widespread approach. However, heuristic algorithms of fuzzy clustering display low level of complexity and high level of essential clarity. Some heuristic clustering algorithms are based on a definition of the cluster concept and the aim of these algorithms is cluster detection conform to a given definition. Such algorithms are called algorithms of direct classification or direct clustering algorithms.

A heuristic approach to possibilistic clustering is proposed in [7]. The essence of the heuristic approach to possibilistic clustering is that the sought clustering structure of the set of observations is formed based directly on the formal definition of fuzzy cluster and possibilistic memberships are determined also directly from the values of the pairwise similarity of observations. A concept of the allotment among fuzzy clusters is basic concept of the approach and the allotment among fuzzy clusters is a special case of the possibilistic partition [2].

### 1.2 Related Works
Since the original Atanassov's [8] paper was published, intuitionistic fuzzy set theory has been applied to many areas and new concepts were introduced. In particular, intuitionistic fuzzy clustering procedures were elaborated by different researchers. Relational and prototype-based intuitionistic fuzzy clustering procedures are existing. The matrix of intuitionistic fuzzy tolerance or intolerance relation is the input data for relational procedures. Let us consider in brief some proposed methods.

In the first place, fuzzy clustering method based on intuitionistic fuzzy tolerance relations was proposed by Hung, Lee and Fuh in [9]. An intuitionistic fuzzy similarity relation matrix is obtained by beginning with an intuitionistic fuzzy tolerance relation matrix using the extended $n$-step procedure by using the composition of intuitionistic fuzzy relations. A hard partition for corresponding thresholds values $\alpha$ and $\beta$ is the result of classification. Several types of $\max\!-\!\mathrm{T}\ \&\ \min\!-\!\mathrm{S}$ compositions can be used in the Hung, Lee and Fuh's approach where $\mathrm{T}$ is some $\mathrm{T}$-norm and $\mathrm{S}$ is a corresponding $\mathrm{S}$-norm.

In the second place, concepts of the association matrix and the equivalent association matrix were defined by Xu, Chen and Wu [10]. Thus, methods for calculating the association coefficients of intuitionistic fuzzy sets were introduced in

[10]. The proposed clustering algorithm uses the association coefficients of intuitionistic fuzzy sets to construct an association matrix, and utilizes a procedure to transform it into an equivalent association matrix. The $\alpha$ -cutting matrix of the equivalent association matrix is used to clustering the given intuitionistic fuzzy sets. So, a hard partition for some value of $\alpha$ is the result of classification.

In the third place, Cai, Lei and Zhao [11] presented a clustering technique based on the intuitionistic fuzzy dissimilarity matrix and $(\alpha, \beta)$ -cutting matrices. The method is based on the transitive closure technique.

In the fourth place, a method to constructing an intuitionistic fuzzy tolerance matrix from a set of intuitionistic fuzzy sets and a netting method to clustering of intuitionistic fuzzy sets via the corresponding intuitionistic fuzzy tolerance matrix are developed by Wang, Xu, Liu and Tang in [12]. A hard partition is the result of classification and the clustering result depends on the chosen value of a confidence level $\alpha \in [0,1]$.

Let us consider some prototype-based intuitionistic fuzzy-set clustering methods. These methods are based on the representation of the initial data by a matrix of attributes. Several of these methods are objective function-based clustering procedures.

In the first place, Pelekis, Iakovidis, Kotsifakos and Kopanakis [13] proposed a variant of the well-known FCM-algorithm that copes with uncertainty and a similarity measure between intuitionistic fuzzy sets, which is appropriately integrated in the clustering algorithm. The ordinary fuzzy $c$ -partition is the clustering result. An application of the proposed clustering technique to image segmentation was described in [14].

In the second place, Torra, Miyamoto, Endo and Domingo-Ferrer [15] proposed a clustering method, based on the FCM-algorithm, to construct an intuitionistic fuzzy partition. In the clustering method, the intuitionistic fuzzy partition deals with the uncertainty present in different executions of the same clustering procedure. Intuitionistic fuzzy partitions for the traditional fuzzy $c$ -means, intuitionistic fuzzy partitions for the entropy-based fuzzy $c$ -means, and intuitionistic fuzzy partitions for the fuzzy $c$ -means with tolerance are considered in [15].

In the third place, an intuitionistic fuzzy approach to distributed fuzzy clustering is considered by Karthikeyani Visalakshi, Thangavel and Parvathi in [16]. The corresponding IFDFC-algorithm is carried out in two different levels: local level and global level. In local level, ordinary numerical data are converted into intuitionistic fuzzy data and the data are clustered independently from each other using the modified FCM-algorithm. In global level, global centroid is calculated by clustering all local cluster centroids and the global centroid is again transmitted to local sites to update the local cluster model.

In the fourth place, a simple clustering technique based on calculating clusters etalons was proposed by Todorova and Vassilev in [17]. The clustering technique based on the assumption that the number of clusters is equal two. The iterative algorithm is stopped when all objects are assigned to crisp clusters according to the preliminary chosen similarity measure.

In the fifth place, agglomerative hierarchical clustering algorithms for classification of ordinary intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets were proposed by Xu in [18]. In each stage of both binary clustering procedures, the center of each cluster should be recalculated by using the average of the intuitionistic fuzzy sets assigned to the cluster, and the distance between two clusters should be determined as the distance between the centers of each cluster.

In the sixth place, an intuitionistic fuzzy $c$ -means method to clustering intuitionistic fuzzy sets is developed by Xu and Wu in [19]. The corresponding IFCM-algorithm assumes that the initial data are represented as a set of intuitionistic fuzzy sets which are defined on the universe of attributes. The method is extended for clustering interval-valued intuitionistic fuzzy sets and the corresponding IVIFCM-algorithm is also described in [19]. The fuzzy $c$ -partition is the clustering result obtained from both algorithms.

In the seventh place, the WIFCM-algorithm based on weighted intuitionistic fuzzy sets is proposed in [20]. The concepts of equivalent classification object and weighted intuitionistic fuzzy set were introduced, and the objective function for the WIFCM-algorithm was derived from these concepts.

In the eighth place, an intuitionistic fuzzy possibilistic $c$ -means algorithm to clustering intuitionistic fuzzy sets is proposed in [21]. The corresponding IFPCM-algorithm is developed by hybridizing concepts of the FPCM clustering method [22], intuitionistic fuzzy sets and distance measures. The IFPCM-algorithm resolves inherent problems encountered with information regarding membership values of objects to each cluster by generalizing membership and non-membership with hesitancy degree. Moreover, the IFPCM-algorithm is extended in [21] for clustering interval-valued intuitionistic fuzzy sets leading to interval-valued intuitionistic fuzzy possibilistic $c$ -means algorithm. So, the IVIFPCM-algorithm has membership and non-membership degrees as intervals.

Different intuitionistic fuzzy-set clustering methods are also described by Xu in [23]. The heuristic approach to possibilistic clustering is generalized for a case of intuitionistic fuzzy tolerance and the corresponding D-PAIFC-algorithm is proposed in [7]. The aim of the presented paper is a consideration of the intuitionistic fuzzy-set prototype-based heuristic algorithm of possibilistic clustering. So, the contents of this paper are the following: in the second section basic definitions of the intuitionistic fuzzy set theory are described and basic concepts of the intuitionistic fuzzy generalization of the heuristic approach to possibilistic clustering are considered, in the third section the formula to derive the intuitionistic fuzzy tolerance degree between intuitionistic fuzzy sets is considered and the D-PAIFC-TC-algorithm is proposed, in the fourth section two illustrative examples are given, in fifth section some preliminary conclusions are formulated and perspectives of future investigations are outlined.

## 2. BASIC DEFINITIONS

The first subsection of the section includes a consideration of basic definitions of the intuitionistic fuzzy set theory. Basic concepts of the intuitionistic fuzzy generalization of the heuristic approach to possibilistic clustering are described in the second subsection.

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## 2.1 Basic Notions of the Intuitionistic Fuzzy Set Theory

Let $X = \{x_1, \ldots, x_n\}$ be a set of elements. An intuitionistic fuzzy set $IA$ in $X$ is given by ordered triple $IA = \{\langle x_i, \mu_{IA}(x_i), \nu_{IA}(x_i)\rangle \mid x_i \in X\}$, where $\mu_{IA}, \nu_{IA} : X \to [0,1]$ should satisfy a condition

$$0 \le \mu_{IA}(x_i) + \nu_{IA}(x_i) \le 1, \qquad (1)$$

for all $x_i \in X$. The values $\mu_{IA}(x_i)$ and $\nu_{IA}(x_i)$ denote the degree of membership and the degree of non-membership of element $x_i \in X$ to $IA$, respectively.

For each intuitionistic fuzzy set $IA$ in $X$ an intuitionistic fuzzy index [8] of an element $x_i \in X$ in $IA$ can be defined as follows

$$\rho_{IA}(x_i) = 1 - \left(\mu_{IA}(x_i) + \nu_{IA}(x_i)\right). \qquad (2)$$

The intuitionistic fuzzy index $\rho_{IA}(x_i)$ can be considered as a hesitancy degree of $x_i$ to $IA$. It is seen that $0 \le \rho_{IA}(x_i) \le 1$ for all $x_i \in X$. Obviously, when $\nu_{IA}(x_i) = 1 - \mu_{IA}(x_i)$ for every $x_i \in X$, the intuitionistic fuzzy set $IA$ is an ordinary fuzzy set in $X$. For each ordinary fuzzy set $A$ in $X$, we have $\rho_A(x_i) = 0$, for all $x_i \in X$.

Let $\mathrm{IFS}(X)$ denote the set of all intuitionistic fuzzy sets in $X$. Basic operations on intuitionistic fuzzy sets were defined by Atanassov in [8] and other publications. In particular, if $IA, IB \in \mathrm{IFS}(X)$ then

$$IA \cap IB = \{\langle x_i, \mu_{IA}(x_i) \wedge \mu_{IB}(x_i), \nu_{IA}(x_i) \vee \nu_{IB}(x_i)\rangle \mid x_i \in X\}, (3)$$

and

$$IA \cup IB = \{\langle x_i, \mu_{IA}(x_i) \vee \mu_{IB}(x_i), \nu_{IA}(x_i) \wedge \nu_{IB}(x_i)\rangle \mid x_i \in X\}. (4)$$

Moreover, some properties of intuitionistic fuzzy sets were given also in [24]. For example, if $IA, IB \in \mathrm{IFS}(X)$, then

$$\begin{array}{l} IA \le IB \Leftrightarrow \mu_{IA}(x_i) \le \mu_{IB}(x_i) \text{ and} \\ \nu_{IA}(x_i) \ge \nu_{IB}(x_i),\ \forall x_i \in X \end{array}, \qquad (5)$$

$$\begin{array}{l} IA \preceq IB \Leftrightarrow \mu_{IA}(x_i) \le \mu_{IB}(x_i) \text{ and} \\ \nu_{IA}(x_i) \le \nu_{IB}(x_i),\ \forall x_i \in X \end{array}, \qquad (6)$$

$$IA = IB \Leftrightarrow IA \le IB \text{ and } IA \ge IB,\ \forall x_i \in X, \quad (7)$$

$$\bar{IA} = \{\langle x_i, \nu_{IA}(x_i), \mu_{IA}(x_i)\rangle \mid x_i \in X\}. \qquad (8)$$

Some definitions will be useful in further considerations. In particular, an $\alpha, \beta$-level of an intuitionistic fuzzy set $IA$ in $X$ can be defined as

$$IA_{\alpha,\beta} = \{x_i \in X \mid \mu_{IA}(x_i) \ge \alpha, \nu_{IA}(x_i) \le \beta\},$$

(9)

where the condition

$$0 \le \alpha + \beta \le 1, \qquad (10)$$

is met for any values $\alpha$ and $\beta$, $\alpha, \beta \in [0,1]$.

The concept of the $(\alpha, \beta)$-level intuitionistic fuzzy set was defined in [7] as follows. The $(\alpha, \beta)$-level intuitionistic fuzzy set $IA_{(\alpha,\beta)}$ in $X$ is given by the following expression:

$$IA_{(\alpha,\beta)} = \left\{ \left\langle \begin{array}{l} x_i \in IA_{\alpha,\beta}, \mu_{IA_{(\alpha,\beta)}}(x_i) = \mu_{IA}(x_i), \\ \nu_{IA_{(\alpha,\beta)}}(x_i) = \nu_{IA}(x_i) \end{array} \right\rangle \right\}, \qquad (11)$$

where $\alpha, \beta \in [0,1]$ should satisfy the condition (10) and $IA_{\alpha,\beta}$ is the $\alpha, \beta$-level of an intuitionistic fuzzy set $IA$ which is satisfied the condition (9).

If $IA$ is an intuitionistic fuzzy set in $X$, where $X$ is the set of elements, then the $(\alpha, \beta)$-level intuitionistic fuzzy set $IA_{(\alpha,\beta)}$ in $X$, for which

$$\mu_{IA_{(\alpha,\beta)}}(x_i) = \begin{cases} \mu_{IA}(x_i), & \text{if } \mu_{IA}(x_i) \ge \alpha \\ 0, & \text{otherwise} \end{cases}, \qquad (12)$$

and

$$\nu_{IA_{(\alpha,\beta)}}(x_i) = \begin{cases} \nu_{IA}(x_i), & \text{if } \nu_{IA}(x_i) \le \beta \\ 0, & \text{otherwise} \end{cases}. \qquad (13)$$

is called an $(\alpha, \beta)$-level intuitionistic fuzzy subset $IA_{(\alpha,\beta)}$ of the intuitionistic fuzzy set $IA$ in $X$ for some $\alpha, \beta \in [0,1]$, $0 \le \alpha + \beta \le 1$. Obviously, that the condition $IA_{(\alpha,\beta)} \preceq IA$ is met for any intuitionistic fuzzy set $IA$ and its $(\alpha, \beta)$-level intuitionistic fuzzy subset $IA_{(\alpha,\beta)}$ for any $\alpha, \beta \in [0,1]$, $0 \le \alpha + \beta \le 1$. The important property will be very useful in further considerations.

Let us remind some basic definitions which were considered by Burillo and Bustince in [24], [25]. In cluster analysis, one is only interested in relations in a set $X$ of classified objects.

Let $X = \{x_1, \ldots, x_n\}$ be an ordinary non-empty set. The binary intuitionistic fuzzy relation $IR$ on $X$ is an intuitionistic fuzzy subset $IR$ of $X \times X$, which is given by the expression

$$IR = \{\langle (x_i, x_j), \mu_A(x_i, x_j), \nu_A(x_i, x_j)\rangle \mid x_i, x_j \in X\}, \qquad (14)$$

where $\mu_{IR} : X \times X \to [0,1]$ and $\nu_{IR} : X \times X \to [0,1]$ satisfy the condition $0 \le \mu_{IR}(x_i, x_j) + \nu_{IR}(x_i, x_j) \le 1$ for every $(x_i, x_j) \in X \times X$.

Let $\mathrm{IFR}(X)$ denote the set of all intuitionistic fuzzy relations on $X$. Let us consider some basic properties of intuitionistic fuzzy relations.

In order to define in the intuitionistic fuzzy relations some properties, the $\mathrm{T}$-norms and $\mathrm{S}$-norms were used in [24]. The $\mathrm{T}$-norms or $\mathrm{S}$-norms defined as a class of intersection or aggregation operators for ordinary fuzzy sets. In particular, for some two ordinary fuzzy sets $A$ and $B$ in $X$ with their membership functions $\mu_A(x_i) \in [0,1]$, $\mu_B(x_i) \in [0,1]$, $\forall x_i \in X$ some typical dual pairs of $\mathrm{T}$-norms $\mathrm{T}_q$ and $\mathrm{S}$-norms $\mathrm{S}_q$, $q \in \{1,2,3\}$ are defined in [26] as follows:

1) $\mathrm{T}_1(\mu_A(x_i), \mu_B(x_i)) = \min(\mu_A(x_i), \mu_B(x_i))$,
$\mathrm{S}_1(\mu_A(x_i), \mu_B(x_i)) = \max(\mu_A(x_i), \mu_B(x_i))$;

2) $\mathrm{T}_2(\mu_A(x_i), \mu_B(x_i)) = \mu_A(x_i) \cdot \mu_B(x_i)$,
$\mathrm{S}_2(\mu_A(x_i), \mu_B(x_i)) = \mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i) \cdot \mu_B(x_i)$;

3) $\mathrm{T}_3(\mu_A(x_i), \mu_B(x_i)) = \max\begin{pmatrix} 0, \\ \mu_A(x_i) + \mu_B(x_i) - 1 \end{pmatrix}$,
$\mathrm{S}_3(\mu_A(x_i), \mu_B(x_i)) = \min(1, \mu_A(x_i) + \mu_B(x_i))$.

Let $\mathrm{A}, \mathrm{B}$ be $\mathrm{T}$-norms and $\Lambda, \mathrm{P}$ be $\mathrm{S}$-norms, and $IR, IQ \in \mathrm{IFR}(X)$ be two binary intuitionistic fuzzy relations on $X$. So, the composed relation $IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ \in \mathrm{IFR}(X)$ to the one is defined by

$$IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ = \left\{ \left\langle \begin{matrix} (x_i, x_k), \mu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k), \\ \nu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k) \end{matrix} \right\rangle \middle| x_i, x_k \in X \right\}, \quad (15)$$

where

$$\mu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k) = \underset{x_j}{\mathrm{A}}\{\mathrm{B}[\mu_{IR}(x_i, x_j), \mu_{IQ}(x_j, x_k)]\}, \quad (16)$$

and

$$\nu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k) = \underset{x_j}{\Lambda}\{\mathrm{P}[\nu_{IR}(x_i, x_j), \nu_{IQ}(x_j, x_k)]\}. \quad (17)$$

The following condition

$$0 \le \mu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k) + \nu_{IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IQ}(x_i, x_k) \le 1, \quad (18)$$

must be met for all $(x_i, x_k) \in X \times X$ in the previous definition.

An intuitionistic fuzzy relation $IR \in \mathrm{IFR}(X)$ is reflexive if for every $x_i \in X$, $\mu_{IR}(x_i, x_i) = 1$ and $\nu_{IR}(x_i, x_i) = 0$. An intuitionistic fuzzy relation $IR \in \mathrm{IFR}(X)$ is called symmetric if for all $(x_i, x_j) \in X \times X$, $\mu_{IR}(x_i, x_j) = \mu_{IR}(x_j, x_i)$ and $\nu_{IR}(x_i, x_j) = \nu_{IR}(x_j, x_i)$. We will say that an intuitionistic fuzzy relation $IR \in \mathrm{IFR}(X)$ is transitive if $IR \ge IR \overset{\mathrm{A,B}}{\underset{\Lambda,\mathrm{P}}{\circ}} IR$. In other words, the intuitionistic fuzzy relation is transitive if condition $\mu_{IR}(x_i, x_k) \ge \underset{x_j \in X}{\vee}\{\mathrm{B}[\mu_{IR}(x_i, x_j), \mu_{IR}(x_j, x_k)]\}$ and condition $\nu_{IR}(x_i, x_k) \le \underset{x_j \in X}{\wedge}\{\mathrm{P}[\nu_{IR}(x_i, x_j), \nu_{IR}(x_j, x_k)]\}$ are met for all $x_i, x_j, x_k \in X$. We will call transitive closure of some intuitionistic fuzzy relation $IR \in \mathrm{IFR}(X)$, to the minimum intuitionistic fuzzy relation $\tilde{IR} \in \mathrm{IFR}(X)$ which contains $IR$ and it is transitive. So, a condition $IR \subseteq \tilde{IR}$ is met.

An intuitionistic fuzzy relation $IT$ in $X$ is called an intuitionistic fuzzy tolerance if it is reflexive and symmetric. An intuitionistic fuzzy relation $IS$ in $X$ is called an intuitionistic fuzzy similarity relation if it is reflexive, symmetric and transitive.

An $n$-step procedure by using the composition of intuitionistic fuzzy relations beginning with an intuitionistic fuzzy tolerance can be used for construction of the transitive closure of an intuitionistic fuzzy tolerance $IT$ and the transitive closure is an intuitionistic fuzzy similarity relation $IS$. The procedure is a basis of the clustering procedure which was proposed by Hung, Lee and Fuh in [9].

An $\alpha, \beta$-level of an intuitionistic fuzzy relation $IR$ in $X$ was defined in [9] as

$$IR_{\alpha,\beta} = \{(x_i, x_j) \mid \mu_R(x_i, x_j) \ge \alpha, \nu_R(x_i, x_j) \le \beta\}, \quad (19)$$

where the condition (10) is met for any values $\alpha$ and $\beta$, $\alpha, \beta \in [0,1]$. So, if $0 \le \alpha_1 \le \alpha_2 \le 1$ and $0 \le \beta_2 \le \beta_1 \le 1$ with $0 \le \alpha_1 + \beta_1 \le 1$ and $0 \le \alpha_2 + \beta_2 \le 1$, then $IR_{\alpha_2,\beta_2} \subseteq IR_{\alpha_1,\beta_1}$. The proposition was formulated in [9].

The $(\alpha, \beta)$-level intuitionistic fuzzy relation $IR_{(\alpha,\beta)}$ in $X$ was defined in [7] as follows:

$$IR_{(\alpha,\beta)} = \left\{ \left\langle \begin{matrix} (x_i, x_j) \in IR_{\alpha,\beta}, \\ \mu_{IR_{(\alpha,\beta)}}(x_i, x_j) = \mu_{IR}(x_i, x_j), \\ \nu_{IR_{(\alpha,\beta)}}(x_i, x_j) = \nu_{IR}(x_i, x_j) \end{matrix} \right\rangle \right\}, \quad (20)$$

where $\alpha, \beta \in [0,1]$ should satisfy the condition (10) and $IR_{\alpha,\beta}$ is the $\alpha, \beta$-level of an intuitionistic fuzzy relation $IR$ which is satisfied the condition (19). The concept of the $(\alpha, \beta)$-level intuitionistic fuzzy relation will be very useful in further considerations.

## 2.2 Basic Concepts of the Intuitionistic Fuzzy Generalization of the Heuristic Approach to Possibilistic Clustering

Let us consider intuitionistic extensions of basic concepts of the D-PAFC-algorithm, which were proposed in [7]. Let $X = \{x_1, \ldots, x_n\}$ be the initial set of elements and $IT$ be some binary intuitionistic fuzzy tolerance on $X = \{x_1, \ldots, x_n\}$ with $\mu_{IT}(x_i, x_j) \in [0,1]$ being its membership function and

$\nu_{IT}(x_i, x_j) \in [0,1]$ being its non-membership function. Let $\alpha$ and $\beta$ be the $\alpha, \beta$-level values of $IT$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $0 \le \alpha + \beta \le 1$. Columns or lines of the intuitionistic fuzzy tolerance matrix are intuitionistic fuzzy sets $\{IA^1, \ldots, IA^n\}$. Let $\{IA^1, \ldots, IA^n\}$ be intuitionistic fuzzy sets on $X$, which are generated by an intuitionistic fuzzy tolerance $IT$. The $(\alpha, \beta)$-level intuitionistic fuzzy set $IA^l_{(\alpha,\beta)} = \left\{ \begin{array}{l} (x_i, \mu_{IA^l}(x_i), \nu_{IA^l}(x_i)) \mid \\ \mu_{IA^l}(x_i) \ge \alpha, \nu_{IA^l}(x_i) \le \beta, x_i \in X \end{array} \right\}$ is intuitionistic fuzzy $(\alpha, \beta)$-cluster or, simply, intuitionistic fuzzy cluster. So $IA^l_{(\alpha,\beta)} \subseteq IA^l$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $IA^l \in \{IA^1, \ldots, IA^n\}$ and $\mu_{li}$ is the membership degree of the element $x_i \in X$ for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $l \in \{1, \ldots, n\}$. On the other hand, $\nu_{li}$ is the non-membership degree of the element $x_i \in X$ for the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$.Value of $\alpha$ is the tolerance threshold of intuitionistic fuzzy clusters elements and value of $\beta$ is the difference threshold of intuitionistic fuzzy clusters elements.

The membership degree of the element $x_i \in X$ for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $0 \le \alpha + \beta \le 1$, $l \in \{1, \ldots, n\}$ can be defined as a

$$\mu_{li} = \begin{cases} \mu_{IA^l}(x_i), & x_i \in IA^l_{\alpha,\beta} \\ 0, & otherwise \end{cases}, \tag{21}$$

where an $\alpha, \beta$-level $IA^l_{\alpha,\beta}$ of an intuitionistic fuzzy set $IA^l$ is the support of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. So, condition $IA^l_{\alpha,\beta} = Supp(IA^l_{(\alpha,\beta)})$ is met for each intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. The membership degree $\mu_{li}$ can be interpreted as a degree of typicality of an element to an intuitionistic fuzzy cluster.

On the other hand, the non-membership degree of the element $x_i \in X$ for an intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $0 \le \alpha + \beta \le 1$, $l \in \{1, \ldots, n\}$ can be defined as a

$$\nu_{li} = \begin{cases} \nu_{IA^l}(x_i), & x_i \in IA^l_{\alpha,\beta} \\ 0, & otherwise \end{cases}. \tag{22}$$

So, the non-membership degree $\nu_{li}$ can be interpreted as a degree of non-typicality of an element to an intuitionistic fuzzy cluster.

In other words, if columns or lines of intuitionistic fuzzy tolerance $IT$ matrix are intuitionistic fuzzy sets $\{IA^1, \ldots, IA^n\}$ on $X$ then intuitionistic fuzzy clusters $\{IA^1_{(\alpha,\beta)}, \ldots, IA^n_{(\alpha,\beta)}\}$ are intuitionistic fuzzy subsets of fuzzy sets $\{IA^1, \ldots, IA^n\}$ for some values $\alpha \in (0,1]$ and $\beta \in [0,1)$, $0 \le \alpha + \beta \le 1$. So, a condition $0 \le \mu_{li} + \nu_{li} \le 1$ is met for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$.

If conditions $\mu_{li} = 0$ and $\nu_{li} = 0$ are met for some element $x_i \in X$ and for some intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$, then the element will be called the residual element of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. The value zero for a fuzzy set membership function is equivalent to non-belonging of an element to a fuzzy set. That is why values of tolerance threshold $\alpha$ are considered in the interval $(0,1]$. So, the value of a membership function of each element of the intuitionistic fuzzy cluster is the degree of similarity of the object to some typical object of fuzzy cluster. From other hand, the value one for an intuitionistic fuzzy set non-membership function is equivalent to non-belonging of an element to an intuitionistic fuzzy set. That is why values of difference threshold $\beta$ are considered in the interval $[0,1)$.

Let $IT$ is an intuitionistic fuzzy tolerance on $X$, where $X$ is the set of elements, and $\{IA^1_{(\alpha,\beta)}, \ldots, IA^n_{(\alpha,\beta)}\}$ is the family of intuitionistic fuzzy clusters for some $\alpha \in (0,1]$ and $\beta \in [0,1)$. The point $\tau^l_e \in IA^l_{\alpha,\beta}$, for which

$$\tau^l_e = \arg\max_{x_i} \mu_{li}, \ \forall x_i \in IA^l_{\alpha,\beta} \tag{23}$$

is called a typical point of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$. Obviously, the membership degree of a typical point of an intuitionistic fuzzy cluster is equal one because an intuitionistic fuzzy tolerance $IT$ is the reflexive intuitionistic fuzzy relation. So, the non-membership degree of a typical point of an intuitionistic fuzzy cluster is equal zero. Moreover, a typical point of an intuitionistic fuzzy cluster does not depend on the value of tolerance threshold and an intuitionistic fuzzy cluster can have several typical points. That is why symbol $e$ is the index of the typical point.

Let $IR^{\alpha,\beta}_z(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} \mid l = \overline{1,c}, c \le n, \\ \alpha \in (0,1], \beta \in [0,1) \end{array} \right\}$ be a family of intuitionistic fuzzy clusters for some value of tolerance threshold $\alpha \in (0,1]$ and some value of difference threshold $\beta \in [0,1)$, $0 \le \alpha + \beta \le 1$. These intuitionistic fuzzy clusters are generated by some intuitionistic fuzzy tolerance $IT$ on the initial set of elements $X = \{x_1, \ldots, x_n\}$. If condition

$$\sum_{l=1}^{c} \mu_{li} > 0, \ \forall x_i \in X \tag{24}$$

and condition

$$\sum_{l=1}^{c} \nu_{li} \ge 0, \ \forall x_i \in X \tag{25}$$

are met for all $IA^l_{(\alpha,\beta)}$, $l = \overline{1,c}$, $c \le n$ then the family is the allotment of elements of the set $X = \{x_1, \ldots, x_n\}$ among intuitionistic fuzzy clusters $\{IA^l_{(\alpha,\beta)}, l=\overline{1,c}, 2 \le c \le n\}$ for some value of the tolerance threshold $\alpha \in (0,1]$ and some value of the difference threshold $\beta \in [0,1)$. It should be noted that several allotments $IR^{\alpha,\beta}_z(X)$ can exist for some pair of thresholds $\alpha$ and $\beta$. That is why symbol $z$ is the index of an allotment.

The condition (24) requires that every object $x_i, i=\overline{1,n}$ must be assigned to at least one intuitionistic fuzzy cluster $IA^l_{(\alpha)}, l=\overline{1,c}, c \le n$ with the membership degree higher than zero and the condition is similar to the definition of the possibilistic partition [7]. The condition $2 \le c \le n$ requires that the number of intuitionistic fuzzy clusters in $IR^{\alpha,\beta}_z(X)$ must be more than two. Otherwise, the unique intuitionistic fuzzy cluster will contain all objects, possibly with different positive membership and non-membership degrees.

The number $c$ of fuzzy clusters can be equal the number of objects, $n$. This is taken into account in further considerations.

Allotment $IR^{\alpha,\beta}_I(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} \mid l=\overline{1,n}, \\ \alpha \in (0,1],\ \beta \in [0,1) \end{array} \right\}$ of the set of objects among $n$ intuitionistic fuzzy clusters for some pair of thresholds $\alpha$ and $\beta$, $0 \le \alpha + \beta \le 1$, is the initial allotment of the set $X=\{x_1,\ldots,x_n\}$. In other words, if initial data are represented by a matrix of some intuitionistic fuzzy tolerance relation $IT$ then lines or columns of the matrix are intuitionistic fuzzy sets $IA^l, l=\overline{1,n}$ and $(\alpha,\beta)$-level fuzzy sets $IA^l_{(\alpha,\beta)}$, $l=\overline{1,n}$, $\alpha \in (0,1]$, $\beta \in [0,1)$ are intuitionistic fuzzy clusters. These intuitionistic fuzzy clusters constitute an initial allotment for some pair of thresholds $\alpha$ and $\beta$ and they can be considered as clustering components.

If condition

$$\bigcup_{l=1}^{c} IA^l_{\alpha,\beta} = X \ , \tag{26}$$

and condition

$$card(IA^l_{\alpha,\beta} \cap IA^m_{\alpha,\beta}) = 0, \forall IA^l_{(\alpha,\beta)}, IA^m_{(\alpha,\beta)}, \\ l \ne m, \alpha,\beta \in (0,1] \ , \tag{27}$$

are met for all intuitionistic fuzzy clusters $IA^l_{(\alpha,\beta)}$, $l=\overline{1,c}$ of some allotment $IR^{\alpha,\beta}_z(X) = \left\{ \begin{array}{l} IA^l_{(\alpha,\beta)} \mid l=\overline{1,c}, c \le n, \\ \alpha \in (0,1],\ \beta \in [0,1) \end{array} \right\}$ then the allotment is the allotment among fully separate intuitionistic fuzzy clusters.

Thus, the problem of cluster analysis can be defined in general as the problem of discovering the unique allotment $IR^*(X)$, resulting from the classification process, which corresponds to either most natural allocation of objects among intuitionistic fuzzy clusters or to the researcher's opinion about classification. In the first case, the number of intuitionistic fuzzy clusters $c$ is not fixed. In the second case, the researcher's opinion determines the kind of the allotment sought and the number of intuitionistic fuzzy clusters $c$ can be fixed. Detection of the unknown least number of compact and well-separated intuitionistic fuzzy clusters can be considered as the aim of classification. For this purpose, the concept of the principal allotment among intuitionistic fuzzy clusters can be introduced as follows.

Allotment $IR^{\alpha,\beta}_P(X) = \{A^l_{(\alpha,\beta)} \mid l=\overline{1,c}, c \le n, \alpha,\beta \in [0,1]\}$ of the set of objects among the minimal number $2 \le c < n$ of fully separate intuitionistic fuzzy clusters for some value of tolerance threshold $\alpha \in (0,1]$ and some value of difference threshold $\beta \in [0,1)$ is the principal allotment of the set $X=\{x_1,\ldots,x_n\}$.

Several principal allotments among intuitionistic fuzzy clusters can exist for some pair of thresholds $\alpha$ and $\beta$. Thus, the problem consists in the selection of the unique principal allotment $IR^*_P(X)$ from the set $B$ of principal allotments, $B=\{IR^{\alpha,\beta}_{P_z}(X)\}$, which is the class of possible solutions of the concrete classification problem. The symbol $z$ is the index of the principal allotments. The selection of the unique principal allotment $IR^*_P(X)$ from the set $B=\{IR^{\alpha,\beta}_{P_z}(X)\}$ of principal allotments must be made on the basis of evaluation of allotments. The criterion

$$F(IR^{\alpha,\beta}_{P_z}(X),\alpha,\beta) = \left( \sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \mu_{li} - \alpha \cdot c \right) - \\ - \left( \sum_{l=1}^{c} \frac{1}{n_l} \sum_{i=1}^{n_l} \nu_{li} - \beta \cdot c \right) \ , \tag{28}$$

where $c$ is the number of intuitionistic fuzzy clusters in the allotment $IR^{\alpha,\beta}_{P_z}(X)$ and $n_l = card(IA^l_{\alpha,\beta})$, $IA^l_{(\alpha,\beta)} \in IR^{\alpha,\beta}_{P_z}(X)$ is the number of elements in the support of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$ can be used for evaluation of allotments. The criterion (28) is the intuitionistic extension of the criterion, which was proposed in [7] for the heuristic D-AFC(c)-algorithm of possibilistic clustering.

Maximum of criterion (28) corresponds to the best allotment of objects among $c$ intuitionistic fuzzy clusters. So, the classification problem can be characterized formally as determination of the solution $IR^*_P(X)$ satisfying

$$IR^*_P(X) = \arg \max_{IR^{\alpha,\beta}_{P_z}(X) \in B} F(IR^{\alpha,\beta}_{P_z}(X),\alpha,\beta) \ , \tag{29}$$

where $B=\{IR^{\alpha,\beta}_{P_z}(X)\}$ is the set of principal allotments corresponding to the pair of thresholds $\alpha$ and $\beta$.

A clustering procedure is based on the decomposition of initial intuitionistic fuzzy tolerance $IT$ [7]. That is why basic concepts of the method of decomposition must be considered. Let $IT$ be an intuitionistic fuzzy tolerance in $X$. Let

$IT_{(\alpha,\beta)}$ be $(\alpha,\beta)$-level intuitionistic fuzzy relation and the condition (10) is met for values $\alpha$ and $\beta$, $\alpha\in(0,1]$, $\beta\in[0,1)$. Let $IT_{\alpha,\beta}$ be a $\alpha,\beta$-level of an intuitionistic fuzzy tolerance $IT$ in $X$ and $IT_{\alpha,\beta}$ be the support of $IT_{(\alpha,\beta)}$. The membership function $\mu_{IT_{(\alpha,\beta)}}(x_i,x_j)$ can be defined as

$$\mu_{IT_{(\alpha,\beta)}}(x_i,x_j)=\begin{cases}\mu_{IT}(x_i,x_j), \text{if } \mu_{IT}(x_i,x_j)\ge\alpha\\ 0, \qquad\qquad \text{otherwise}\end{cases}, \qquad (30)$$

and the non-membership function $\nu_{IT_{(\alpha,\beta)}}(x_i,x_j)$ can be defined as

$$\nu_{IT_{(\alpha,\beta)}}(x_i,x_j)=\begin{cases}\nu_{IT}(x_i,x_j), \text{if } \nu_{IT}(x_i,x_j)\le\beta\\ 0, \qquad\qquad \text{otherwise}\end{cases}. \qquad (31)$$

Obviously, that the condition $IT_{(\alpha,\beta)}\preceq IT$ is met for any intuitionistic fuzzy tolerance $IT$ and a $(\alpha,\beta)$-level intuitionistic fuzzy relation $IT_{(\alpha,\beta)}$ for any $\alpha\in(0,1]$, $\beta\in[0,1)$, $0\le\alpha+\beta\le1$. Thus we have the proposition that if $\alpha_{\ell(\alpha)}\le\alpha_{\ell+1(\alpha)}$ and $\beta_{\ell+1(\beta)}\le\beta_{\ell(\beta)}$ with $0\le\alpha_{\ell(\alpha)}+\beta_{\ell(\beta)}\le1$, $0\le\alpha_{\ell+1(\alpha)}+\beta_{\ell+1(\beta)}\le1$ then the condition $IT_{(\alpha_{\ell+1(\alpha)},\beta_{\ell+1(\beta)})}\preceq IT_{(\alpha_{\ell(\alpha)},\beta_{\ell(\beta)})}$ is met. So, the ordered sequences $0<\alpha_0\le\ldots\le\alpha_{\ell(\alpha)}\le\ldots\le\alpha_{Z(\alpha)}\le1$ and $0\le\beta_{Z(\beta)}\le\ldots\le\beta_{\ell(\beta)}\le\ldots\le\beta_0<1$ must be constructed for the decomposition of an intuitionistic fuzzy tolerance $IT$. A method of construction of sequences was developed in [7].

## 3. AN OUTLINE OF THE METHOD

A method to derive the intuitionistic fuzzy tolerance degree between intuitionistic fuzzy sets is considered in the first subsection. The prototype-based clustering D-PAIFC-TC-algorithm is proposed in the second subsection of the section.

### 3.1 On Constructing the Intuitionistic Fuzzy Tolerance Relation

The method for constructing the intuitionistic fuzzy tolerance relation was proposed by Wang, Xu, Liu and Tang in [12]. The corresponding similarity measure is based on the normalized Hamming distance and the similarity measure can be expressed by a formula

$$r(IA,IB)=\begin{cases}(1,\,0), & IA=IB\\ \begin{pmatrix}1-\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left|\nu_{IA}(x_i)-\nu_{IB}(x_i)\right|-\\ -\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left|\rho_{IA}(x_i)-\rho_{IB}(x_i)\right|,\\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\left|\nu_{IA}(x_i)-\nu_{IB}(x_i)\right|\end{pmatrix}, & IA\ne IB\end{cases}, \qquad (32)$$

for all $i,j=1,\ldots,n$. That is why the closeness degree $r(IA,IB)=\left(\mu_{IT}(IA,IB),\nu_{IT}(IA,IB)\right)$ of intuitionistic fuzzy sets $IA$ and $IB$ can be constructed according to the formula (32). Obviously, if all the differences of values of the non-membership degree and the differences of values of the intuitionistic fuzzy index of two objects $IA$ and $IB$ with respect to attributes $x_i$, $i=1,\ldots,n$ get smaller, then the two objects are more similar to each other.

The corresponding intuitionistic fuzzy relation possesses the symmetry property and the reflexivity property. Moreover, the condition $0\le\mu_{IT}(IA,IB)+\nu_{IT}(IA,IB)\le1$ is met for any intuitionistic fuzzy sets $IA$ and $IB$. These facts were proved in [12].

### 3.2 A Plan of the D-PAIFC-TC-Algorithm

The principal allotment $IR_P^{\alpha,\beta}(X)$ is a family of intuitionistic fuzzy clusters which are elements of the initial allotment $IR_I^{\alpha,\beta}(X)$ for the pair of thresholds $\alpha$ and $\beta$ and the family of intuitionistic fuzzy clusters should satisfy the conditions (26) and (27). So, the construction of principal allotments $IR_{P_z}^{\alpha,\beta}(X)=\left\{\begin{matrix}A^l_{(\alpha,\beta)}\mid l=\overline{1,c}, c\le n,\\ \alpha\in(0,1],\ \beta\in[0,1)\end{matrix}\right\}$ for every pair of thresholds $\alpha$ and $\beta$ is a trivial combinatorial problem.

Each object can be represented as an intuitionistic fuzzy set $x_i$, $i=1,\ldots,n$ on the universe of $m_1$ attributes and $x_i^{t_1}=(\mu_{x_i}(x^{t_1}),\nu_{x_i}(x^{t_1}))$, $i=1,\ldots,n$, $t_1=1,\ldots,m_1$ are their membership and non- membership functions. So, an initial data set can be represented as an intuitionistic matrix of attributes, $X_{n\times m_1}=[x_i^{t_1}=(\mu_{x_i}(x^{t_1}),\nu_{x_i}(x^{t_1}))]$.

Thus, the D-PAIFC-TC-algorithm is a ten-step procedure of classification:

1. Construct the matrix of the intuitionistic fuzzy tolerance $IT_{n\times n}=[\mu_{IT}(x_i,x_j),\nu_{IT}(x_i,x_j)]$ in $X$ with respect to the formula (32);

2. Construct the transitive closure $\tilde{IT}$ of the intuitionistic fuzzy tolerance $IT$ according to selected $\mathrm{T}$-norm $\mathrm{T}_q$ and $\mathrm{S}$-norm $\mathrm{S}_q$, $q\in\{1,2,3\}$;

3. Construct ordered sequences $0<\alpha_0\le\ldots\le\alpha_{\ell(\alpha)}\le\ldots\le\alpha_{Z(\alpha)}\le1$ and $0\le\beta_{Z(\beta)}\le\ldots\le\beta_{\ell(\beta)}\le\ldots\le\beta_0<1$ of thresholds values; let $\ell(\alpha):=0$ and $\ell(\beta):=0$;

4. The following condition is checked:

   **if** the condition $0\le\alpha_{\ell(\alpha)}+\beta_{\ell(\beta)}\le1$ is met

   **then** construct the $(\alpha,\beta)$-level intuitionistic fuzzy relation $\tilde{IT}_{(\alpha,\beta)}$, which satisfy condition (20) and go to step 5

   **else** the following condition is checked:

   **if** the condition $\ell(\beta)<Z(\beta)$ is met

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**then** let $\ell(\beta) := \ell(\beta)+1$ and go to step 4;

5. Construct the initial allotment $IR_I^{\alpha,\beta}(X) = \{A^l_{(\alpha,\beta)} \mid l=\overline{1,n}, \alpha \in (0,1], \beta \in [0,1)\}$ for calculated values $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$;
6. The following condition is checked:

   **if** for some intuitionistic fuzzy cluster $A^l_{(\alpha,\beta)} \in IR_I^{\alpha,\beta}(X)$ the condition $card(A^l_{(\alpha,\beta)}) = n$ is met

   **then** let $\ell(\beta) := \ell(\beta)+1$ and go to step 4

   **else** construct the allotments, which satisfy conditions (26) and (27);
7. The following condition is checked:

   **if** for $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$ allotments $IR_{P_z}^{\alpha,\beta}(X)$ satisfying conditions (26) and (27) are not constructed and the condition $\ell(\beta) < Z(\beta)$ is met

   **then** let $\ell(\beta) := \ell(\beta)+1$ and go to step 4

   **else** **if** the condition $\ell(\beta) = Z(\beta)$ is met

   **then** $\ell(\alpha) := \ell(\alpha)+1$ and go to step 4

   **else** construct the class of possible solutions of the classification problem $B = \{IR_{P_z}^{\alpha,\beta}(X)\}$ for $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$;
8. The following condition is checked:

   **if** for some unique allotment $IR_{P_z}^{\alpha,\beta}(X) \in B$ the condition (29) is met

   **then** the allotment is the classification result $IR_P^*(X)$ for corresponding values $\alpha_{\ell(\alpha)}$ and $\beta_{\ell(\beta)}$ and stop

   **else** construct the set of allotments $B' \subseteq B$ which satisfy the condition (29) and go to step 9;
9. Perform the following operations for each allotment $IR_{P_z}^{\alpha,\beta}(X) \in B$:

   9.1 Let $l := 1$;

   9.2 Determine the support $Supp(IA^l_{(\alpha,\beta)}) = IA^l_{\alpha,\beta}$ of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)} \in IR_{P_z}^{\alpha,\beta}(X)$ and construct the intuitionistic matrix of attributes $X_{n_l \times m_1} = [x_i^{t_1} = (\mu_{x_i}(x^{t_1}), \nu_{x_i}(x^{t_1}))]$, $x_i \in IA^l_{\alpha,\beta}$, $t_1 = 1,\ldots,m_1$, for $IA^l_{\alpha,\beta}$, where $n_l = card(IA^l_{\alpha,\beta})$;

   9.3 Calculate the prototype $\tau^l = \{x^1,\ldots,x^{m_1}\}$, $x^{t_1} = \left(\mu_{x_i}(x^{t_1}), \nu_{x_i}(x^{t_1})\right)$ of the class $IA^l_{\alpha,\beta}$ according to the formulae

   $$\mu_{x_i}(x^{t_1}) = \frac{1}{n_l}\sum_{x_i \in A^l_{\alpha,\beta}} \mu_{x_i}(x^{t_1}),\ t_1 = 1,\ldots,m_1,$$

   $$\nu_{x_i}(x^{t_1}) = \frac{1}{n_l}\sum_{x_i \in A^l_{\alpha,\beta}} \nu_{x_i}(x^{t_1}),\ t_1 = 1,\ldots,m_1;$$

   9.4 Calculate the similarity measure $r(\tau^l, \tau^l)$ (32) between the typical point $\tau^l$ of the intuitionistic fuzzy cluster $IA^l_{(\alpha,\beta)}$ and its prototype $\tau^l$;

   9.5 The following condition is checked:

   **if** all intuitionistic fuzzy clusters $IA^l_{(\alpha,\beta)} \in IR_{P_z}^{\alpha,\beta}(X)$ are not verified

   **then** let $l := l+1$ and go to step 9.2

   **else** go to step 10;
10. Compare the intuitionistic fuzzy clusters $IA^l_{(\alpha,\beta)}$ which are the elements of different allotments $IR_{P_z}^{\alpha,\beta}(X) \in B'$, and the allotment for which the similarity measure $r(\tau^l, \tau^l)$ (32) is maximal for all intuitionistic fuzzy clusters $IA^l_{(\alpha,\beta)}$ is the classification result $IR_P^*(X)$.

The principal allotment $IR_P^*(X) = \{IA^l_{(\alpha,\beta)} \mid l = \overline{1,c}\}$ among unknown number $c$ of fully separate intuitionistic fuzzy clusters, prototypes $\{\tau^1,\ldots,\tau^c\}$ of the corresponding intuitionistic fuzzy clusters, the value of tolerance threshold $\alpha \in (0,1]$ and the value of difference threshold $\beta \in [0,1)$ are results of classification.

# 4. ILLUSTRATIVE EXAMPLES

The performance of the proposed D-PAIFC-TC-algorithm is explained by two Wang's [12] illustrative examples which are given in both subsection of the section.

## 4.1 The First Example

Let us consider the first Wang's [12] illustrative example. An auto market requires classifying five different cars $x_i$, $i = 1,\ldots,5$ into several kinds. Each car has six evaluation factors: $x^1$ – oil consumption, $x^2$ – coefficient of friction, $x^3$ – price, $x^4$ – comfortable degree, $x^5$ – design, $x^6$ – safety coefficient. The evaluation results of each car with respect to the factors $x^{t_1}$, $t_1 = 1,\ldots,6$ are represented by the intuitionistic fuzzy sets, shown in Table 1. By applying the formula (32) to the set $\{x_i \mid i = \overline{1,5}\}$ of intuitionistic fuzzy sets, a matrix of intuitionistic fuzzy tolerance relation was obtained.

**Table 1. The characteristics information of the cars**

| Cars | Factors | | | | | |
|---|---|---|---|---|---|---|
| | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ |
| $x_1$ | (0.3, 0.5) | (0.6, 0.1) | (0.4, 0.3) | (0.8, 0.1) | (0.1, 0.6) | (0.5, 0.4) |
| $x_2$ | (0.6, 0.3) | (0.5, 0.2) | (0.6, 0.1) | (0.7, 0.1) | (0.3, 0.6) | (0.4, 0.3) |
| $x_3$ | (0.4, 0.4) | (0.8, 0.1) | (0.5, 0.1) | (0.6, 0.2) | (0.4, 0.5) | (0.3, 0.2) |
| $x_4$ | (0.2, 0.4) | (0.4, 0.1) | (0.9, 0.0) | (0.8, 0.1) | (0.2, 0.5) | (0.7, 0.1) |
| $x_5$ | (0.5, 0.2) | (0.3, 0.6) | (0.6, 0.3) | (0.7, 0.1) | (0.6, 0.2) | (0.5, 0.3) |

By executing the D-PAIFC-TC-algorithm, the principal allotment $IR_P^*(X)$ among two intuitionistic fuzzy clusters was obtained. Membership functions and non-membership functions of two classes are presented in Figure 1.

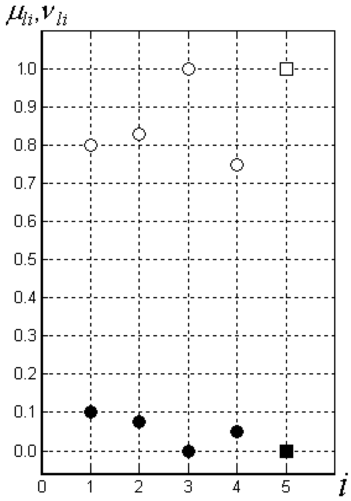


**Fig 1: The membership values and non-membership values of two intuitionistic fuzzy clusters**

The allotment was received for the tolerance threshold $\alpha = 0.68$ and the value of difference threshold $\beta = 0.18$.

Membership values of the first class are represented by ○, non-membership values of the first class are represented by ●, membership values of the second class are represented by □ and non-membership values of the second class are represented by ■ in Figure 1. The third object is the typical point of the first intuitionistic fuzzy cluster and the fifth object is the typical point of the second intuitionistic fuzzy cluster.

On the other hand, the hard partition $\{x_1, x_2, x_3, x_4\}$, $\{x_5\}$ was obtained in [12] for $0.68 < \alpha \le 0.72$.

## 4.2 The Second Example

Let us consider the second Wang's [12] illustrative example. The data set contains the information of ten new cars $x_i$, $i = 1, \ldots, 10$ to be classified into an auto market. Each car is described by six attributes: $x^1$ – oil consumption, $x^2$ – coefficient of friction, $x^3$ – price, $x^4$ – comfortable degree, $x^5$ – design, $x^6$ – safety coefficient, as in the first example. The characteristics of cars under the six factors $x^{t_1}$, $t_1 = 1, \ldots, 6$ are represented by the intuitionistic fuzzy sets, as shown in Table 2.

**Table 2. The characteristics of ten cars**

| Cars | Factors | | | | | |
|---|---|---|---|---|---|---|
| | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ |
| $x_1$ | (0.8, 0.1) | (0.4, 0.1) | (0.6, 0.1) | (0.7, 0.3) | (0.6, 0.2) | (0.5, 0.0) |
| $x_2$ | (0.0, 0.3) | (0.1, 0.3) | (0.0, 0.6) | (0.0, 0.5) | (0.5, 0.3) | (0.4, 0.2) |
| $x_3$ | (0.2, 0.0) | (0.9, 0.1) | (0.0, 0.7) | (0.0, 0.1) | (0.3, 0.2) | (0.8, 0.2) |
| $x_4$ | (0.0, 0.5) | (0.3, 0.0) | (0.7, 0.1) | (0.6, 0.1) | (0.0, 0.7) | (0.7, 0.2) |
| $x_5$ | (0.4, 0.6) | (0.2, 0.4) | (0.9, 0.1) | (0.6, 0.1) | (0.7, 0.2) | (0.7, 0.3) |
| $x_6$ | (0.0,0.2) | (0.0,0.0) | (0.5,0.4) | (0.5,0.4) | (0.3,0.6) | (0.0,0.0) |
| $x_7$ | (0.8, 0.1) | (0.2, 0.1) | (0.1, 0.0) | (0.7, 0.0) | (0.6, 0.4) | (0.0, 0.6) |
| $x_8$ | (0.1, 0.7) | (0.0, 0.5) | (0.8, 0.1) | (0.7, 0.1) | (0.7, 0.1) | (0.0, 0.0) |
| $x_9$ | (0.0, 0.1) | (0.5, 0.1) | (0.3, 0.1) | (0.7, 0.3) | (0.1, 0.3) | (0.7, 0.2) |
| $x_{10}$ | (0.3, 0.2) | (0.7, 0.1) | (0.2, 0.2) | (0.2, 0.0) | (0.1, 0.9) | (0.9, 0.1) |

By applying the D-PAIFC-TC-algorithm to the data, the principal allotment $IR_P^*(X)$ among four intuitionistic fuzzy clusters, which corresponds to the result, is received for the tolerance threshold $\alpha = 0.48$ and the value of difference threshold $\beta = 0.18$. Values of membership functions and non-membership functions of four classes are presented in Figure 2, where membership values of the first class are represented by ○, non-membership values of the first class are represented by ●, membership values of the second class are represented by □, non-membership values of the second class are represented by ■, membership values of the third class are represented by ∇, non-membership values of the third class are represented by ▼, membership values of the fourth class are represented by Δ and non-membership values of the fourth class are represented by ▲.

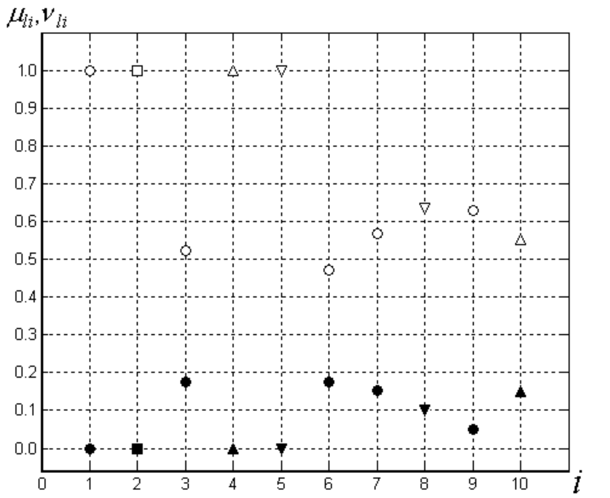


**Fig 2: The membership values and non-membership values of two intuitionistic fuzzy clusters**

The first object is the typical point of the first intuitionistic fuzzy cluster, the second object is the typical point of the second intuitionistic fuzzy cluster, the fifth object is the typical point of the third intuitionistic fuzzy cluster and the fourth object is the typical point of the fourth intuitionistic fuzzy cluster.

It should be noted, that the hard partition into five classes $\{x_1, x_5\}$, $\{x_2, x_4, x_6, x_9, x_{10}\}$, $\{x_3\}$, $\{x_7\}$, $\{x_8\}$ was obtained in [12] for the value of confidence level $0.55 < \alpha \leq 0.57$.

## 5. CONCLUDING REMARKS

Results of experiments are summarized and discussed in the first subsection of the section. The second subsection deals with the perspectives on future investigations.

### 5.1 Discussions

The intuitionistic heuristic prototype-based D-PAIFC-TC-algorithm of possibilistic clustering is proposed in the paper. The notion of the principal allotment of the data set among an unknown number of intuitionistic fuzzy clusters is a conceptual basis of the proposed algorithm. Moreover, the matrix of attributes as the matrix of the initial data and the Wang's [12] similarity measure are used in the D-PAIFC-TC-algorithm of possibilistic clustering. A procedure for constructing transitive closure of an intuitionistic fuzzy tolerance relation is also used in the D-PAIFC-TC-algorithm.

The D-PAIFC-TC-algorithm was tested on two illustrative data sets. The results of application of the proposed algorithm to these data sets show that the algorithm is the effective tool for solving the classification problem under ambiguity of the initial data.

### 5.2 Perspectives

Let us consider some prospective ways for further investigations.

In the first place, constructing of the allotment among unknown number $c$ of fully separate intuitionistic fuzzy clusters can be considered as an aim of classification. For the purpose, an idea of a leap in similarity values for finding the appropriate value $\alpha_{\ell(\alpha)}$ of the tolerance threshold and a leap in dissimilarity values for finding the appropriate value $\beta_{\ell(\beta)}$ of the difference threshold can be useful. So, the leap heuristic [7] should be generalized for the case of intuitionistic fuzzy set-based clustering.

In the second place, an algorithm for constructing the allotment among unknown number $c$ of fully separate intuitionistic fuzzy clusters with respect to the given minimal value $\alpha$ of tolerance threshold and the given maximal value $\beta$ of difference threshold.

In the third place, the proposed algorithm can be re-considered in the framework of the MIVAR-oriented way in artificial intelligence [27], [28].

These perspectives for investigations are of great interest both from the theoretical point of view and from the practical one as well.

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