Perfect Line Domination in Graphs

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ABSTRACT
Here we are defining perfect line domination set and some results on perfect line domination.

Keywords
Line domination, perfect line domination, neighborhood, minimal line domination

1. INTRODUCTION
Definition 1.1: Perfect line dominating set:
A subset $S$ of $E(G)$ is said to be perfect line dominating set if for each edge not in $S$, $e$ is adjacent to exactly one edge of $S$.

Consider the path $P_3$ with 4 vertices and edge are $\{ e_1, e_2, e_3 \}$. The set $\{ e_2 \}$ is perfect line dominating set in this graph.

It may be noted that If $G$ is a graph then $E(G)$ is always a perfect line dominating set.

Definition 1.2: Minimal perfect line dominating set:
A perfect line dominating set $S$ of the graph $G$ is said to be minimal perfect line dominating set if each line $e$ in $S$, $S-\{ e \}$ is not a perfect line dominating set.

It may be noted that it is not necessary that a proper subset of minimal perfect line dominating set is not a perfect line dominating set.

Example 1.3: Consider the cycle graph $G = C_3$ with 5 vertices. Then obviously $E(G)$ is a minimal perfect line dominating set of $G$.

However the set $\{ e_1, e_2 \}$ is proper subset of $E(G)$ and is a perfect line dominating set in the graph $G$.

Definition 1.4: Minimum perfect line dominating set:
A perfect line dominating set $S$ with smallest cardinality is called set of the graph minimum perfect line dominating set it is called $\gamma_{pf}(G)$.

Definition 1.5: Perfect line domination number:
The cardinality of a minimum perfect line dominating set is called the perfect domination number of the graph $G$. It is denoted by $\gamma_{pf}(G)$.

The perfect line domination number of cycle $C_6$ is 2. $C_3$ is 2 and that of path $P_4$ is 1.

Definition 1.6: perfect private line neighborhood:
Let $S$ be the subset of $E(G)$ and $e \in S$ then the perfect private neighborhood of $e$ with respect to $S = P_{pf}[e, S] = \{ x \in E(G)-S : N(x) \cap \{ e \} \cup \{ e \} \text{ if } e \text{ is adjacent to no line of } S \text{ or at least line of } S \}.$

Theorem 1.7: a perfect line dominating set $S$ of $G$ is minimal perfect line dominating set if and only if for each line $e$ in $S$ $P_{pf}[e, S]$ is nonempty.

Suppose $S$ is minimal and $e \in S$. Therefore there is a line $x$ not in $S$ \{ $e$ \} or $x$ is adjacent to at least two lines of $S - \{ e \}$.

If $x = e$ then this implies that $e \in P_{pf}[e, S]$.

If $x \neq e$ then it is impossible that $x$ is adjacent to at least two lines of $S - \{ e \}$. Therefore $S$ is a perfect line dominating set.

Conversely, suppose $e \in S$ and $P_{pf}[e, S]$ contains some line $x$ of $G$.

If $x = e$ then $x$ is adjacent to at least two lines of $S - \{ e \}$.

If $x \neq e$ then $N(x) \cap S = \{} e \}$ implies that $x$ is not adjacent to any line of $S - \{ e \}$.

Thus in all cases $S - \{ e \}$ is not a perfect line dominating set, if $e \in S$. Thus $S$ is minimal.

Example 1.8: Consider the path $G = P_3$ with five lines $e_1, e_2, e_3, e_4, e_5$. Note that $S = \{ e_2, e_4 \}$ is minimum and therefore minimal perfect line dominating set $P_{pf}[e_2, S] = \{ e_1, e_2 \}$.

We define the following symbols,

$E^+_{pf} = \{ e \in E(G) : \gamma_{pf}(G) \leq \gamma_{pf}(G-e) \}$

$E^-_{pf} = \{ e \in E(G) : \gamma_{pf}(G) > \gamma_{pf}(G-e) \}$

$E^0_{pf} = \{ e \in E(G) : \gamma_{pf}(G) = \gamma_{pf}(G-e) \}$

Remark: The above sets are mutually disjoint and their union is $E(G)$.

Lemma 1.9: Let $e \in E(G)$ and suppose $e$ is a pendant line and has a neighbor $x$ if $e \in E^-_{pf}$ then

$\gamma_{pf}(G-e) = \gamma_{pf}(G) - 1$.

Let $S_1$ be a minimum line dominating set of $G$.

$E - \{ e \}$ if $x \in S_1$ then $S_1$ is a perfect line dominating set of $G$ with $|S_1| < \gamma_{pf}(G)$.

i.e., $\gamma_{pf}(G) = |S_1| \leq \gamma_{pf}(G-e)$ this is a contradiction, therefore $\gamma_{pf}(G) = |S_1| = |S_1| + 1 = \gamma_{pf}(G-e) + 1$.

Hence the lemma.

Theorem 1.10: Let $e$ be a line of $G$ Then $e \in E^+_{pf}$ if and only if the following conditions are satisfied, i) $e$ belongs to every $\gamma_{pf}$ set of $G$, ii) no subset $s$ of $G - \{ e \}$ which is either disjoint from $N[e]$ or intersects $N[e]$ in at least two lines and $|S| \leq \gamma_{pf}$ can be perfectly dominating set of

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G – {e}.

Proof: (i) Suppose e ∈ E_{pf}^+.

Suppose S is a \( \gamma_{pf} \) set of G which does not contains e then S is a perfect dominating set of G – {e}.

Therefore \( \gamma_{pf} (G – e) \leq |S| = \gamma_{pf} (G) \), thus e ∉ E_{pf}^+. This is a contradiction. Then e must belong to every \( \gamma_{pf} \) set of G.

(ii) If there is set S which satisfies the condition stated in (ii) then S is a perfect line dominating set of G – {e} and therefore \( \gamma_{pf} (G – e) \leq \gamma_{pf} (G) \) This is a contradiction.

Conversely, assume that (i) and (ii) hold

Suppose e ∈ E_{pf}^0. Let s be a minimum perfect line dominating set of G – {e}.

Then |S| = \( \gamma_{pf} (G) \).

Suppose e is not adjacent to any line of S. Then S is disjoint from N[e]. |S| = \( \gamma_{pf} (G) \). and S is perfectly line dominating set of G – {e}. This violates (ii)

Suppose e is adjacent to exactly one line of S then S is minimum perfect line dominating set of G not containing e which violates (i)

Suppose e is adjacent to at least two lines of S, then |S| ∩ N[e] in at least two lines and S is perfectly line dominating set of G – {e} with |S| = \( \gamma_{pf} (G) \) which again violates (ii).

Thus e ∈ E_{pf}^0 implies (i) and (ii) violated.

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3. REFERENCES


