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A Relation between Laplace and Generalized Hankel-Clifford Transformation

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ABSTRACT
A relation between the Laplace transform and the generalized Hankel-Clifford transform is obtained. An attempt has been made to establish the relation between distributional generalized Hankel-Clifford transform and distributional one sided Laplace transform. The results are verified by giving illustrations.

Keywords
Generalized Hankel-Clifford transforms, Laplace Transform, Generalized Function, Testing Function, inversion theorem.

1. INTRODUCTION
Schwartz first introduced the Fourier transform of distributions in 1947. Since then, extension of the classical integral transformation to generalized functions has been of continuing interest. In Zemanian [9], Laplace transformations of distributions can be studied. Finite Hankel and Hankel type transform of classical functions were first introduced by Sneddon [8]. Malgonde and Lakshmi Gorty [10] extended generalized Hankel-Clifford transforms to certain spaces of distributions as a special case of the general theory on orthonormal series expansions of generalized functions. In [2, 6], the authors have shown that, for any positive arbitrary value, the transform of Kekre function is obtained and how Kekre's function is related to inverse Laplace transforms. The generalized representation of the Laplace transforms of Kekre's function is also formulated. Author in paper [6] defined a finite generalized Laplace Hankel-Clifford transformation of a certain generalized functions, and established an inversion formula. It is quite well known that there are several problems which can be solved by the repeated applications of the transformations. If an integral transforms constructed for which the kernel is the product of the kernels of the Laplace and Generalized Hankel-Clifford transformation of the first kind, integral (special) transform as Laplace Hankel-Clifford transform which is successfully applied to deal with the problems occurring in mathematical physics shown in [7].

2. PRELIMINARY RESULTS
The Laplace transform of a function of a function $f(t) \in L(0,\infty)$ is defined as:

$$L(f;p) = \int_0^\infty e^{-pt} f(t)\,dt;\ (\mathrm{Re}(p) > 0) \qquad (1)$$

and Malgonde [4] investigated the variant of the generalized Hankel-Clifford transform defined by

$$(h_{\alpha,\beta} f)(\xi) = F(\xi)$$
$$= \int_0^\infty (\xi/t)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{t\xi}\right) f(t)\,dt\ ,\ (\alpha-\beta) \geq -1/2$$
$$= \xi^{-\alpha-\beta} \int_0^\infty \mathcal{J}_{\alpha,\beta}(t\xi)\, f(t)\,dt\ ,\ (\alpha-\beta) \geq -1/2 \qquad (2)$$

where $\mathcal{J}_{\alpha,\beta}(t) = (t)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{t}\right)$, $J_{\alpha-\beta}(t)$, being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions. Note that (2) reduces to well-known Hankel-Clifford transform for suitable values of the parameters viz. for $\alpha = 0$ and $\beta = -\mu$, a transform studied in [4].

A relation between the Laplace transform of $t^{\alpha+\beta} f(t)$ and the generalized Hankel-Clifford transform of $f(t)$, when $(\mathrm{Re}(\alpha-\beta) > -1)$. The result is stated in the form of a theorem which is then illustrated by an example.

Theorem 1: If f and $(h_{\alpha,\beta} f)(\xi)$ belongs to $L(0,\infty)$ and if $\mathrm{Re}(a) > 0, \mathrm{Re}(p) > 0, \mathrm{Re}(\alpha+\beta) > -1;$ then $L\left\{t^{\alpha+\beta} f(t); p\right\} = \int_0^\infty k(p,\xi)(h_{\alpha,\beta} f)(\xi)\,d\xi$, where $k(p,\xi)$ $= p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha, 1+\alpha-\beta, -\frac{\xi}{p}\right]$.

Proof. Since $f \in L(0,\infty)$, by the generalized Hankel-Clifford inversion theorem [4], that

$$f(t) = \int_0^\infty \xi^{\alpha+\beta} (h_{\alpha,\beta} f)(\xi)\, \mathcal{J}_{\alpha,\beta}(t\xi)\,d\xi \cdot$$

Hence

$$L\left\{t^{\alpha+\beta} f(t)\right\}$$
$$= \int_0^\infty \xi^{\alpha+\beta} (h_{\alpha,\beta} f)(\xi)\, L\left\{t^{\alpha+\beta} \mathcal{J}_{\alpha,\beta}(t\xi)\right\} d\xi \cdot$$

The change of order of integration is justified because $e^{-pt} t^{\alpha+\beta} \in L(0,\infty)$ if $\mathrm{Re}(\alpha-\beta) > -1;\ \mathrm{Re}(p) > 0$ and $(h_{\alpha,\beta} f)(\xi) \in L(0,\infty)$; $\mathcal{J}_{\alpha,\beta}(t\xi)$ being a bounded function of both the variables. The theorem then follows from the fact [8] that

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$$L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi);p\right\} = p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{p}\right]. \quad (3)$$

Example 1: Let $f(t)=t^{n-1}e^{-at}$. Then

$$L\left\{t^{\alpha+\beta}f(t);p\right\}=(p+a)^{-n-\alpha-\beta}\,\Gamma(\alpha+\beta+n) \quad (4)$$

and

$$\begin{aligned}(h_{\alpha,\beta}f)(\xi)&=F(\xi)\\ &=\xi^{-\alpha-\beta}\int_0^\infty e^{-at}t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\,dt\\ &=\xi^{-\alpha-\beta}L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi);a\right\}.\end{aligned} \quad (5)$$

This last integral can be evaluated by using (3). Substituting these expressions in the theorem analogous to [5]; the result is

$$\int_0^\infty \xi^{-\alpha-\beta}\left(h_{\alpha,\beta}f\right)(\xi)L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\right\}d\xi$$

$$=\int_0^\infty \xi^{-\alpha}\xi^{-\beta}L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi);a\right\}\;L\left\{t^{\alpha+\beta}\mathcal{J}_{\alpha,\beta}(t\xi)\right\}d\xi$$

$$=\int_0^\infty p^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{p}\right]\times a^{-1-\alpha}\xi^{\alpha}\Gamma(1+\alpha)\,{}_1F_1\left[1+\alpha,1+\alpha-\beta,-\frac{\xi}{a}\right]d\xi$$

$$=\frac{(p+a)^{-n-\alpha-\beta}\,\Gamma(\alpha+\beta+n)}{\Gamma(1+\alpha+n)\Gamma(1+2\alpha+\beta)}$$

where

$$a>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(p)>0,\ \mathrm{Re}(\alpha-\beta)>-\frac{1}{2};$$

$$\mathrm{Re}(\alpha)>-\frac{1}{2},\ \mathrm{Re}(\alpha+n)>-1.$$

3. RELATION BETWEEN TRANSFORMS TO THE SPACE OF DISTRIBUTIONS

Let $(h_{\alpha,\beta}f)(\xi)$ is a testing function space for generalized Hankel Clifford transform and $(h'_{\alpha,\beta}f)(\xi)$ is its dual. $L(w,z)$ and $L(w)$ are testing function spaces for Laplace transform and $L'(w,z)$ and $L'(w)$ are their duals respectively. Since the testing function space $(h_{\alpha,\beta}f)(\xi)$, $L(w,z)$ and $L(w)$ are subspace of E, the space of distributions of compact support E' is a subspace of all the generalized function space $(h'_{\alpha,\beta}f)(\xi)$, $L'(w,z)$ and $L'(w)$.

The restriction $f\in(h'_{\alpha,\beta}f)(\xi)\cap L'(w,z)$ to E is a member of E' [4]. In order to extend the relation (3) to the space of distributions, a lemma is stated.

Lemma 1 If $f\in L'(w,z)$ then the mapping $f(x)\to x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)f(x)$ is a linear and continuous from $L'(w)$ into itself.

Proof: For each integer $k\geq 0$ there exists an integer n_k such that

$$\left|(1+x^2)^{N_k}D^k\left[x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)\right]\right|$$

$$=\left|(1+x^2)^{N_k}\right|\left|\sum_{m=0}^{k}\binom{k}{m}D^{k-m}D^m\left[x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)\right]\right|$$

$$=\left|(1+x^2)^{N_k}\right|\left|\sum_{m=0}^{k}\binom{k}{m}D^{k-m}\times\left[\sum_{j=0}^{m}a_j(\alpha)\,y^{-\left(\frac{\alpha+\beta+j}{2}\right)}\lambda_n^{\,j-m}\times x^{\left(\frac{\alpha+\beta+j}{2}\right)}J_{\alpha-\beta-j}\left(2\sqrt{x\lambda_n}\right)\right]\right|$$

$$=\left|(1+x^2)^{N_k}\right|\left|\sum_{m=0}^{k}\binom{k}{m}\left[\sum_{j=0}^{m}a_j(\alpha)\lambda_n^{-\left(\frac{\alpha+\beta+j}{2}\right)}\lambda_n^{\,j-m}\times x^{\left(\frac{\alpha+\beta+j}{2}\right)}J_{\alpha-\beta-j}\left(2\sqrt{x\lambda_n}\right)\times(-1)^{k-m}x^{(\alpha+\beta+k-m)/2}\lambda_n^{-(\alpha-\beta-j+k-m)/2}\times J_{\alpha-\beta-j+k-m}\left(2\sqrt{x\lambda_n}\right)\right]\right|$$

where $0<x<\infty$ and where the $a_j(\alpha)$ are constants depending on α only as in [1].

Therefore $x^{-(\alpha-\beta)/2}J_{\alpha-\beta}\left(2(\lambda_n x)^{1/2}\right)\in\theta_M$.

As $e^{-px}L(w,z)$, the mapping $e^{-px}\to x^{-(\alpha-\beta)/2}J_{\alpha-\beta}\left(2(\lambda_n x)^{1/2}\right)e^{-px}$ is linear and continuous from $L(w,z)$ into itself and the adjoint mapping $f(x)\to x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)f(x)$ defined by

$$\left\langle x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)f(x),e^{-px}\right\rangle=\left\langle f(x),x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(x\lambda_n)e^{-px}\right\rangle \quad (6)$$

is a linear and continuous from $L'(w,z)$ into itself [3] [Theorem 1.10 and Sec. 2.5], where

$f\in L'(w,z), e^{-px}\in L(w,z),\ x^{-\alpha-\beta}J_{\alpha,\beta}(x\lambda_n)\in\theta_M$.

Lemma 2. Let $f\in(h'_{\alpha,\beta}f)(\xi)$ then mapping

$f(x) \to e^{-px} f(x)$ is a linear and continuous $(h'_{\alpha,\beta} f)(\xi)$ into itself.

Proof: Since for each nonnegative integer k there exists an integer N_k such that

$\left| \frac{(x^{-\alpha} D)^k e^{-px}}{1 + x^{N_k/2}} \right| < \infty$ for $0 < x < \infty$. Thus

$e^{-px} \in \theta_M$ the space of multipliers for $(h_{\alpha,\beta} f)(\xi)$

[4]. As $e^{-px} \in \theta_M$, the mapping

$x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \to e^{-px} x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n)$

is a linear and continuous from $(h_{\alpha,\beta} f)(\xi)$ into itself and the adjoint mapping $f(x) \to e^{-px} f(x)$, $f \in (h'_{\alpha,\beta} f)(\xi)$ defined by

$$\left\langle e^{-px} f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle = \left\langle f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) e^{-px} \right\rangle \quad (7)$$

is a linear and continuous from $(h'_{\alpha,\beta} f)(\xi)$ into itself, where $f \in (h'_{\alpha,\beta} f)(\xi),\ x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \in (h_{\alpha,\beta} f)(\xi).$

Theorem 2. If $f \in E'$; *then*

$\left\langle e^{-px} f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle = \left\langle x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px} \right\rangle$

for $\mathrm{Re}(p) < \infty, \lambda_n \geq 0; \alpha - \beta \geq -\frac{1}{2}$.

Proof: Since the testing function space $(h_{\alpha,\beta} f)(\xi)$, $L(w,z)$ and $L(w)$ are subspace of E , the space of distributions of compact support E' is a subspace of all the generalized function space $(h'_{\alpha,\beta} f)(\xi), L'(w,z)$ and $L'(w)$ [4]. Therefore the restriction of $f \in L'(w)$ to $L(w,z)$ is in $L'(w,z)$ and the restriction of $f \in (h'_{\alpha,\beta} f)(\xi) \cap L'(w,z)$ to E is a member of E' . In view of above the result is obvious from Lemma 1 and Lemma 2 for every $f \in E'$, since the right hand sides of the equations (6) and (7) are equal. Therefore the equality becomes

$$\left\langle e^{-px} f(x), x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) \right\rangle = \left\langle x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(x\lambda_n) f(x), e^{-px} \right\rangle$$

holds good for $0 < \mathrm{Re}(p) < \infty,\ \lambda_n \geq 0; \alpha - \beta \geq -\frac{1}{2}$.

4. CONCLUSION

Relation between Laplace and generalized Hankel-Clifford transform is established. A testing function space for generalized Hankel-Clifford transform and is its dual is established in this paper. The testing function spaces for Laplace transform and their duals are derived. There are examples given in the text and Lemma have been established. These relations can be applied to many applications in physics and electronics.

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